



A Novel Similarity Measure Based on Belief and Plausibility for Pythagorean Fuzzy Multi-Criteria Decision-Making

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ARTICLE INFO

Article history:

Received 6 August 2026

Received in revised form 1 October 2026

Accepted 9 October 2026

Available online 11 October 2026

Keywords:

Pythagorean Fuzzy Sets; Belief and Plausibility Measures; Linguistic Variables; Multi-Criteria Decision-Making; Belief and Plausibility; Technique for Order Preference by Similarity to Ideal Solution

ABSTRACT

Several generalizations of the Dempster-Shafer theory to fuzzy sets and intuitionistic fuzzy sets have been reported in the literature. However, no studies have extended Dempster-Shafer theory to Pythagorean fuzzy distance and Pythagorean fuzzy similarity measures. This paper proposes novel distance and similarity measures for Pythagorean fuzzy sets within the framework of belief and plausibility functions. They are core components of the Dempster-Shafer theory for managing uncertain, vague, partial, and imprecise information. The validity and applicability of the proposed measures are illustrated through numerical examples, including applications to pattern recognition and linguistic variable analysis. Building on these measures, we further develop a belief- and plausibility-based technique for order preference by similarity to ideal solution (TOPSIS) method by integrating them into TOPSIS. To assess its performance, belief- and plausibility-based TOPSIS is applied to real-world multi-criteria decision-making problems. The results confirm the robustness, practicality, and effectiveness of the proposed method within the Dempster-Shafer theory framework.

1. Introduction

The Dempster-Shafer Theory (DST) has proven to be highly useful in the field of artificial intelligence [1]. It represents a rational approach to reasoning under uncertainty, commonly referred to as evidential reasoning. However, DST has certain limitations, particularly in expert systems. The theory struggles to address complex, wide-ranging concepts effectively. Knowledge based on such indefinite concepts often lacks clarity, especially when it involves the qualitative description of variables without well-defined boundaries. To address this, there is a need to aggregate evidence related to these concepts and manage imbalanced hypotheses. The primary function of DST is to handle imprecise evidential strengths by utilizing belief and plausibility functions, which serve as lower and upper bounds of probability, respectively [1]. However, earlier extensions of DST have not fully preserved its foundational ideas. Shafer's theory aims to simplify probability theory, making it more accessible for understanding complex problems [2].

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Zadeh [3] introduced fuzzy set (FS) theory as a generalization of crisp sets to address uncertainties beyond the scope of DST. FSs are characterized by their membership and non-membership grades. While FS theory effectively handles many types of uncertainty, certain situations arise where it falls short. To address these limitations, the concept of intuitionistic fuzzy sets (IFSs) was proposed [4] by extending FSs to better manage more complex uncertainties. IFSs are interpreted as ordered triplets (μ, ν, π) , where μ denotes the membership grade, ν represents the non-membership grade, and π shows the grade of hesitancy, with the condition that $\mu + \nu + \pi = 1$. However, IFSs are limited in handling scenarios where the sum of the membership and non-membership grades exceeds one. To address this limitation, Yager [5] extended and generalized the concept of IFSs by proposing Pythagorean fuzzy sets (PyFSs), which are characterized by the condition $0 \leq \mu^2 + \nu^2 \leq 1$. Mathematically, PyFSs are characterized as: μ for membership grade, ν for non-membership grade and π for grade of hesitancy in such a way that $\mu^2 + \nu^2 + \pi^2 = 1$ or $0 \leq \mu^2 + \nu^2 \leq 1$, where $\pi^2 = 1 - \mu^2 - \nu^2$. Yager [5,6] introduced several aggregation operations for PyFSs based on these characteristics. To support multi-criteria decision-making (MCDM), he developed a method for comparing Pythagorean membership grades. PyFSs have since become a pivotal tool in MCDM [7]. Additionally, Garg [8] combined the concepts of PyFSs and linguistic FSs to create a novel linguistic PyFS, further expanding their applications.

1.1 Literature Review

Besides probability theory, another emerging field to manage uncertainties is belief and plausibility in the context of DST. They have been widely used in many other fields across a wide range. The generalization of probability theory is the basic purpose of Shafer theory. Shafer [2] introduced belief and plausibility functions on ordinary subsets in the framework of Dempster theory.

DST is a paradigm for reasoning under uncertainty initiated by Dempster [9]. It was formalized by Shafer [2], who extended classical probability theory to address uncertainty, unclear, or incomplete information. DST uses a frame of discernment in which all subsets, not just single hypotheses, are assigned a belief based on basic probability assignments. The representation of incomplete and overlapping evidence is possible using this method. Plausibility and belief functions quantify the degree to which the evidence does not contradict a subset and the overall level of support for it, respectively. By combining data from several sources, Dempster's rule of combination manages disputes and strengthens evidence that is consistent. Applications for DST include risk assessment, information retrieval, sensor fusion, and decision support. These areas all involve complicated decision-making situations where uncertainty is present and determining prior probabilities can be challenging.

On the other hand, FSs and their extensions are proven to be very useful and reasonable tools to handle all types of uncertainties close to reality. The interpretation of IFS to DST is done by Dymova & Sevastjanov [10,11]. Some belief and plausibility functions on IFSs are presented by Hwang & Yang [12] and Yang *et al.* [13]. However, Yang *et al.* [13] did not give consideration to generalized belief and plausibility measures on PyFSs. So, we consider a new belief and plausibility distance and similarity measure (SM) on PyFSs in the context of DST, with its application to MCDM with the novel BIPI (belief-plausibility)-technique for order preference by similarity to ideal solution (TOPSIS). Distance measures are widely used to measure the degree of difference. SMs are used to find the degree of similarity between two different objects or sets [14]. Cheng & Xiao [15] proposed a new distance measure for belief functions. Khalaj & Khalaj [16,17] suggested cosine similarity on belief theory. Hussain *et al.* [18] proposed PyFS similarities under the Jaccard index.

MCDM is a process that could be applied in selecting from a given set of alternatives when there is more than one criterion to be considered at the same time [8,19]. It systematically assesses different options by taking into account a range of quantitative and qualitative factors, each of varying importance. MCDM utilizes various methods to combine these criteria into a comprehensive evaluation, allowing decision-makers to make informed choices by effectively balancing trade-offs among the factors [20,21]. This approach is essential in fields such as engineering, finance, management, public policy, and medical sciences, where decisions must reflect a complex interplay of different aspects [22,23].

1.2 Research Gap and Limitations

Despite the significant advancements in distance and SMs under belief and plausibility functions, their integration within the framework of PyFSs remains unexplored in the literature. This gap highlights the absence of robust tools to assess relationships between PyFSs under conditions of uncertainty and imprecision. Furthermore, existing methodologies are inadequate for addressing and analyzing multiple criteria in complex decision-making scenarios, especially in the presence of ambiguous or incomplete information. Most notably, no attempts have been made to generalize the DST to PyFSs, leaving a critical void in the theoretical and practical applications of DST for managing higher-dimensional fuzzy data. This unexplored area presents an opportunity to advance both the theoretical framework and practical utility of DST within the domain of PyFSs.

Some limitations of existing literature are as follows: They can be computationally complex to handle belief and plausibility functions in large-scale issues. There is a lack of thorough investigation and validation across a range of applications in the DST framework's integration of distance and SMs on PyFSs. Furthermore, the literature does not capture every detail of uncertainty in real-world situations through the use of particular distance and SMs, like cosine similarity or Jaccard index.

1.3 Contributions

- i. We propose new distance and SMs for belief and plausible PyFSs, offering a more precise method for quantifying relationships within PyFSs.
- ii. In a pioneering approach, we apply these proposed SMs to pattern recognition and linguistic variables, demonstrating their practicality and effectiveness in real-world applications.
- iii. To enhance MCDM processes, we develop the BIPI-TOPSIS method by leveraging the proposed measures. This method provides decision-makers with a robust tool to evaluate and compare complex alternatives more effectively and efficiently.

The structure of this paper is as follows: Section 2 explains the integration of PyFSs within the framework of DST. An intuitive formulation of belief and plausibility functions for PyFSs is proposed, which is then utilized to develop a distance measure between two belief and plausibility functions (BIPIs). All axioms are rigorously proven, and numerical examples are presented to validate the proposed measures. Section 3 introduces several SMs based on the proposed belief and plausibility functions, utilizing simple and appropriate mathematical formulations. An example involving linguistic hedges is provided to demonstrate the effectiveness of these SMs between two BIPIs. Section 4 focuses on the application of TOPSIS to construct a belief and plausibility-based decision-making framework (BIPI-TOPSIS). This method is designed to address complex real-world problems involving multi-criteria decision-making. Section 5 concludes the paper with a summary of findings and potential directions for future research.

2. Novel Measures of Belief and Plausibility on Pythagorean Fuzzy Sets

Here, we analyzed PyFSs within the framework of DST, introducing new belief and plausibility functions. When a set M is considered along with its complement M^c within the framework of DST, the belief degree on these two sets and their subsets will be $Bl(M) + Bl(M^c) \leq 1$. In contrast to probability theory, the degree of belief assigned to a statement and its complement may contribute to supporting either proposition. In the event of complete ignorance, then $Bl(M) = Bl(M^c) = 0$ is even possible, whereas the theory of probability in the identical condition of complete ignorance shows $Pro(M) = Pro(M^c) = 0.5$. Because of this condition, the belief function $Bl(M)$ does not accurately describe the belief committed to proposition M . The doubt committed to the exactly same proposition is not described as the belief that is committed to M^c . So, the doubt function $Dou(M) = Bl(M^c)$ is defined to gain a full description of the belief committed M . Salicone [24,25] also discussed the doubt function.

2.1 Interpretation of DST to PyFSs

To interpret DST within the framework of PyFSs, we align the concepts of belief and plausibility with membership degree and non-membership degree. It has been clearly established that DST and PyFS are now intimately related, and they also have some relationships among themselves.

- i. *Belief function and membership degree* – The belief function $Bl_M(x)$ in DST represents the lower bound of probability for an event, analogous to the membership degree $\mu(x)$ in PyFS for every $x \in M$, indicating degree of support for membership based on evidence. Mathematically, it can be expressed as $Bl_M(x) = \mu_M^2(x)$, which reflects that the degree of belonging and degree of belief are interlinked and connected by reflecting the similar demonstration about certain happening.
- ii. *Plausibility function and non-membership degree* – The plausibility function $Pl_M(x)$ in DST provides an upper bound of probability for an event, which in PyFS corresponds to the complement of the non-membership grade $\nu(x)$. Hence, non-membership grade in PyFS represents the degree of support for non-membership, derived inversely from plausibility. A whole explanation of an event M is assumed by belief function $Bl_M(x)$ and a plausible function $Pl_M(x)$, which may be represented mathematically as $Pl_M(x) = 1 - \nu_M^2(x)$.

Theorem 1 [13]. *The following properties satisfy the requirements of Bl and Pl measures on PyFSs:*

- i. $0 \leq Bl_M(x_i) \leq Pl_M(x_i) \leq 1$;
- ii. If $M_1 \subseteq M_2$ then $Bl_{M_1}(x_i) \leq Bl_{M_2}(x_i)$ and $Pl_{M_1}(x_i) \leq Pl_{M_2}(x_i)$;
- iii. $Bl_M(x_i) = 1 - Pl_{M^c}(x_i)$ and $Pl_M(x_i) = 1 - Bl_{M^c}(x_i)$;
- iv. $Bl_M(x_i) + Bl_{M^c}(x_i) \leq 1$ and $Pl_M(x_i) + Pl_{M^c}(x_i) \geq 1$.

Now, we develop belief and plausibility distance measures of PyFSs M and N based on the proposed BI and PI as:

$$d(BIPI_M, BIPI_N) = \frac{1}{2n} \sum_{i=1}^n \{ |BI_M(x_i) - BI_N(x_i)| + |PI_M(x_i) - PI_N(x_i)| \} \quad (1)$$

Besides, the weighted belief and plausibility distance measure of PyFS M and N based on the proposed BI and PI measures will be:

$$d(BIPI_M, BIPI_N) = \frac{1}{2n} \sum_{i=1}^n w_i \{ |BI_M(x_i) - BI_N(x_i)| + |PI_M(x_i) - PI_N(x_i)| \} \quad (2)$$

This distance will be used to construct SMs between BI and PI .

Definition 1 [13]. Consider $BIPI_M$, $BIPI_N$, and $BIPI_O$ be any three BIPISs of PyFS in X . A distance d in Eq. (1) is called the distance between two BIPISs on PyFSs if the below axioms are fulfilled:

- D_1 : (Non negative) $0 \leq d(BIPI_M, BIPI_N) \leq 1$
- D_2 : (Separable) $d(BIPI_M, BIPI_N) = 0$, if and only if $BIPI_M = BIPI_N$
- D_3 : (Symmetric) $d(BIPI_M, BIPI_N) = d(BIPI_N, BIPI_M)$
- D_4 : (Triangular inequality) $d(BIPI_M, BIPI_O) \leq d(BIPI_M, BIPI_N) + d(BIPI_N, BIPI_O)$

Is the proposed distance $d(BIPI_M, BIPI_N)$ is suitable or not is a common question. So, to assess the validity and reasonableness of the suggested distance in Eq. (1), Theorem 2 is provided.

Theorem 2. Consider $X = \{x_1, x_2, \dots, x_n\}$ is a universal set. The suggested distance measure $d(BIPI_M, BIPI_N)$ among two belief and plausibility sets M and N of PyFSs fulfills the properties of Definition 1.

Example 1. Consider the finite universal of set be $X = \{x_1, x_2, \dots, x_n\}$. Consider M be any belief and plausibility set of PyFS in the finite universe X with $M = \{\langle x_1, 0.6, 0.1 \rangle, \langle x_2, 0.5, 0.8 \rangle, \langle x_3, 0.2, 0.7 \rangle\}$. The weights for the set are $w = \{0.7, 0.1, 0.2\}$. We calculate the belief function of PyFSs as given as $BI_M(x_1) = (0.6)^2$, $BI_M(x_2) = (0.5)^2$, $BI_M(x_3) = (0.2)^2$. Similarly, we can calculate the plausibility function as $PI_M(x_1) = 0.99$, $PI_M(x_2) = 0.36$, $PI_M(x_3) = 0.51$. Thus, we calculate the belief function as $BI(M) = \sum_{i=1}^n w_i BI_M(x_i) = (0.7)(0.6)^2 + (0.1)(0.5)^2 + (0.2)(0.2)^2 = 0.285$. Similarly, the plausibility function is $PI(M) = \sum_{i=1}^n w_i PI_M(x_i) = (0.7)(0.99) + (0.1)(0.36) + (0.2)(0.51) = 0.831$. Hence, from the above result it is clear that $BI(M) \leq PI(M)$.

Example 2. Suppose M and N are two PyFSs in X with $M = \{\langle x_1, 0.6, 0.8 \rangle, \langle x_2, 0.5, 0.7 \rangle, \langle x_3, 0.4, 0.9 \rangle\}$ and $N = \{\langle x_1, 0.8, 0.4 \rangle, \langle x_2, 0.5, 0.7 \rangle, \langle x_3, 0.7, 0.6 \rangle\}$. The corresponding belief and plausible PyFSs $BIPI_M$ and $BIPI_N$ of PyFSs M and N can be constructed as

$BIP_{I_M} = \{\langle x_1, 0.6, 0.36 \rangle, \langle x_2, 0.5, 0.51 \rangle, \langle x_3, 0.4, 0.19 \rangle\}$ and $BIP_{I_N} = \{\langle x_1, 0.8, 0.84 \rangle, \langle x_2, 0.5, 0.51 \rangle, \langle x_3, 0.7, 0.64 \rangle\}$.
Using the proposed distance measures in Eq. (1), we have $d(BIP_{I_M}, BIP_{I_N}) = 0.2383$.

3. Similarity Measures between Belief and Plausible Pythagorean Fuzzy Sets

SMs have a key role in differentiating two sets or objects with numerous applications in various fields. They help in the recognition of similar fuzzy patterns in pattern recognition. They make it easier to assign data points to clusters with different degrees of membership. Furthermore, SMs help with risk assessment by comparing fuzzy risk scenarios, improve machine learning models by handling fuzzy data, and improve medical diagnosis by comparing patient symptoms with known diseases. Since distance and SMs are two different concepts, we use the distance between two BIPISs of PyFSs to show their similarities. Assume a monotonically decreasing function f and BIP_{I_M} and BIP_{I_N} be two BIPISs, where $0 \leq d(BIP_{I_M}, BIP_{I_N}) \leq 1$ and $f(1) \leq f(d(BIP_{I_M}, BIP_{I_N})) \leq f(0)$. It suggests $0 \leq \left(\frac{f(d(BIP_{I_M}, BIP_{I_N})) - f(1)}{f(0) - f(1)} \right) \leq 1$. As a result, the similarity between BIP_{I_M} and BIP_{I_N} may be described as:

$$F(BIP_{I_M}, BIP_{I_N}) = \frac{f(d(BIP_{I_M}, BIP_{I_N})) - f(1)}{f(0) - f(1)} \quad (3)$$

Using Eq. (3), various similarities can be obtained by selecting a simple, appropriate, and suitable function f . Let us choose a linear function $f(x) = 1 - x$ to construct new SMs that lie on the distance between two belief and plausibility sets of PyFSs. The new SM between two belief and plausibility sets of PyFSs utilizing Eq. (3) is stated as:

$$S_L(BIP_{I_M}, BIP_{I_N}) = 1 - d(BIP_{I_M}, BIP_{I_N}) \quad (4)$$

Next, we choose $f(x) = \left(\frac{1}{1+x} \right)$ and construct another similarity of belief and plausibility sets of PyFSs using Eq. (3) as follows:

$$S_Q(BIP_{I_M}, BIP_{I_N}) = \frac{1 - d(BIP_{I_M}, BIP_{I_N})}{1 + d(BIP_{I_M}, BIP_{I_N})} \quad (5)$$

Finally, a famous simple and reasonable function $f(x) = e^{-x}$ is chosen to develop a new SM between two belief and plausible sets of PyFSs using Eq. (3) and is given as:

$$S_E(BIP_{I_M}, BIP_{I_N}) = \frac{e^{-d(BIP_{I_M}, BIP_{I_N})} - e^{-1}}{1 - e^{-1}} \quad (6)$$

3.1 Application of the Proposed Similarity Measures in Pattern Recognition

Pattern recognition refers to the study of how machines identify and classify various patterns. It involves using algorithms, machine learning techniques, or human observation to detect and extract meaningful patterns, trends, or anomalies from complex data sets. Here, we show various pattern

recognition examples by using the suggested SMs of Eqs. (4)-(6) to test the reasonableness, validity, and efficiency.

Example 3. In the realm of solar energy research, accurately classifying materials used in solar plates is crucial for optimizing their efficiency and effectiveness. Assume that we have a dataset of material samples. Each sample is characterized by unique properties such as chemical composition, crystal structure, physical attributes, hardness, and optical reluctance. To classify these samples, we can express the three material samples by three PyFSs M_1 , M_2 , and M_3 as $M_1 = \{(x_1, 0.5, 0.5), (x_2, 0.4, 0.4), (x_3, 0.7, 0.7)\}$, $M_2 = \{(x_1, 0.6, 0.6), (x_2, 0.7, 0.6), (x_3, 0.5, 0.6)\}$, and $M_3 = \{(x_1, 0.3, 0.8), (x_2, 0.4, 0.9), (x_3, 0.5, 0.6)\}$. Assume that a sample $N = \{(x_1, 0.5, 0.5), (x_2, 0.4, 0.4), (x_3, 0.7, 0.7)\}$ of another hybrid material is given. The aim is to develop a pattern recognition system which automatically classifies and categorizes material samples according to their attributes and shows which dataset the sample is a part of. The corresponding BIP_{M_1} , BIP_{M_2} , BIP_{M_3} , and BIP_N of M_1 , M_2 , M_3 , and N are as follows:

- $BIP_{M_1} = \{(x_1, 0.25, 0.75), (x_2, 0.16, 0.84), (x_3, 0.49, 0.51)\}$,
- $BIP_{M_2} = \{(x_1, 0.36, 0.64), (x_2, 0.49, 0.64), (x_3, 0.25, 0.64)\}$,
- $BIP_{M_3} = \{(x_1, 0.09, 0.36), (x_2, 0.16, 0.19), (x_3, 0.25, 0.64)\}$, and
- $BIP_N = \{(x_1, 0.25, 0.75), (x_2, 0.16, 0.84), (x_3, 0.49, 0.51)\}$.

By using Eqs. (4)-(6), we have $S_L(BIP_{M_1}, BIP_N) = 1$, $S_Q(BIP_{M_1}, BIP_N) = 1$, $S_E(BIP_{M_1}, BIP_N) = 1$, $S_L(BIP_{M_2}, BIP_N) = 0.813$, $S_Q(BIP_{M_2}, BIP_N) = 0.685$, $S_E(BIP_{M_2}, BIP_N) = 0.731$, $S_L(BIP_{M_3}, BIP_N) = 0.738$, $S_Q(BIP_{M_3}, BIP_N) = 0.585$, $S_E(BIP_{M_3}, BIP_N) = 0.636$. From the calculations, the sample BIP_N matches with pattern BIP_{M_1} conferring to the rules of the highest grade of similarities of two belief and plausible sets. This shows that the sets M_1 and N closely resemble each other and thus match the true state. According to the principle of highest similarity between two BIPISs of PyFSs, the results shows the resemblance of image sample BIP_N with BIP_{M_1} .

3.2 Comparative Analysis of Existing and Proposed Similarity Measures

To test the validity and suitability of our proposed measures of Eqs. (4)-(6), we compare with the SM of Li *et al.* [23], which is given as:

$$Y(\mathbb{X}_1, \mathbb{X}_2) = \frac{3k - (|\sigma_{\mathbb{X}_1}^2 - \sigma_{\mathbb{X}_2}^2| + |\delta_{\mathbb{X}_1}^2 - \delta_{\mathbb{X}_2}^2| + |\eta_{\mathbb{X}_1}^2 - \eta_{\mathbb{X}_2}^2|)}{3k + (|\sigma_{\mathbb{X}_1}^2 - \sigma_{\mathbb{X}_2}^2| + |\delta_{\mathbb{X}_1}^2 - \delta_{\mathbb{X}_2}^2| + |\eta_{\mathbb{X}_1}^2 - \eta_{\mathbb{X}_2}^2|)} \quad (7)$$

where $(|\sigma_{\mathbb{X}_1}^2 - \sigma_{\mathbb{X}_2}^2| + |\delta_{\mathbb{X}_1}^2 - \delta_{\mathbb{X}_2}^2| + |\eta_{\mathbb{X}_1}^2 - \eta_{\mathbb{X}_2}^2|) = \sum_{j=1}^n (|\sigma_{\mathbb{X}_1}^2(s_j) - \sigma_{\mathbb{X}_2}^2(s_j)| + |\delta_{\mathbb{X}_1}^2(s_j) - \delta_{\mathbb{X}_2}^2(s_j)| + |\eta_{\mathbb{X}_1}^2(s_j) - \eta_{\mathbb{X}_2}^2(s_j)|)$,

for feature space $S = (s_1, s_2, \dots, s_k)$, and k is a positive integer.

Example 4. The patterns are $M_1 = \{(x_1, 0.4, 0.7), (x_2, 0.6, 0.2), (x_3, 0.8, 0.5)\}$, $M_2 = \{(x_1, 0.5, 0.6), (x_2, 0.6, 0.7), (x_3, 0.8, 0.3)\}$, $M_3 = \{(x_1, 0.3, 0.9), (x_2, 0.5, 0.6), (x_3, 0.8, 0.5)\}$, and $M_4 = \{(x_1, 0.2, 0.9), (x_2, 0.6, 0.6), (x_3, 0.7, 0.5)\}$. Consider a sample N is given as

$N = \{(x_1, 0.3, 0.9), (x_2, 0.5, 0.6), (x_3, 0.9, 0.4)\}$. The aim is to interpret the notion of given patterns and to show which pattern the image sample N matches. The comparison analysis is given as:

$S_L(BIPI_{M_1}, BIPI_N) = 0.820$	$S_Q(BIPI_{M_1}, BIPI_N) = 0.695$	$S_E(BIPI_{M_1}, BIPI_N) = 0.739$	$S_{given}(M_1, N) = 0.698$
$S_L(BIPI_{M_2}, BIPI_N) = 0.818$	$S_Q(BIPI_{M_2}, BIPI_N) = 0.693$	$S_E(BIPI_{M_2}, BIPI_N) = 0.737$	$S_{given}(M_2, N) = 0.661$
$S_L(BIPI_{M_3}, BIPI_N) = 0.957$	$S_Q(BIPI_{M_3}, BIPI_N) = 0.917$	$S_E(BIPI_{M_3}, BIPI_N) = 0.933$	$S_{given}(M_3, N) = 0.932$
$S_L(BIPI_{M_4}, BIPI_N) = 0.902$	$S_Q(BIPI_{M_4}, BIPI_N) = 0.821$	$S_E(BIPI_{M_4}, BIPI_N) = 0.852$	$S_{given}(M_4, N) = 0.815$

As demonstrated by the suggested similarities of Eqs. (4)–(6), the findings show that the sample N resembles the pattern M_3 in accordance with the maximum degree of similarity between two Pythagorean fuzzy patterns. This confirms that the proposed measures are more appropriate than the current ones.

3.3 Application of the Proposed Similarity Measures in Linguistic Variable Analysis

In this subsection, we give the concept of linguistic terms such as dilation (DIL) and concentration (CON) of $BIPI_M$ corresponding to PyFS M as:

Definition 2 [13]. Consider $N = \{(x, \mu_N(x)), \nu_N(x) \mid x \in X\}$ be an PyFS on X . If n is any constructive real number, the PyFS N^n is described as $N^n = \left\{ \left(\{x, \mu_N^2(x)\}^n, \{1 - \nu_N^2(x)\}^n : x \in X \right) \right\}$. The corresponding $BIPI_N$ may be described as:

$$(BIPI_N)^n = \left\{ \left\langle \{BI_N(x)\}^n, \{PI_N(x)\}^n \right\rangle : x \in X \right\}, \quad n > 0 \quad (8)$$

where $BI_N(x) = \mu_N^2(x)$ and $PI_N(x) = 1 - \nu_N^2(x)$.

If we apply Definition 2, the concentration of $BIPI_N$ can be shown as:

$$CON(BIPI_N) = \left\{ \left\langle x, \{BI_{CON(BIPI_N)}(x), PI_{CON(BIPI_N)}(x)\} \right\rangle : x \in X \right\} \quad (9)$$

where $BI_{CON(BIPI_N)}(x) = (BI_{BIPI_N}(x))^2$ and $PI_{CON(BIPI_N)}(x) = (PI_{BIPI_N}(x))^2$.

Besides, if we apply Definition 2, the dilation of $BIPI_N$ can be shown as:

$$DIL(BIPI_N) = \left\{ \left\langle x, \{BI_{DIL(BIPI_N)}(x), PI_{DIL(BIPI_N)}(x)\} \right\rangle : x \in X \right\} \quad (10)$$

where $BI_{DIL(BIPI_N)}(x) = (BI_{BIPI_N}(x))^{\frac{1}{2}}$ and $PI_{DIL(BIPI_N)}(x) = (PI_{BIPI_N}(x))^{\frac{1}{2}}$.

Example 5. Consider a PyFS M in the universal set X . Then $N = \{\langle x_1, 0.5, 0.7 \rangle, \langle x_2, 0.6, 0.5 \rangle, \langle x_3, 0.6, 0.3 \rangle, \langle x_4, 0.9, 0.2 \rangle\}$. By considering the classification of variables in language, N is regarded as "GREATER" in X . Using concentration and dilation in the context of belief and plausibility, $BIPI_N^{\frac{1}{2}}$ will be regarded as "More or Less GREATER", $BIPI_N^2$ will be regarded as "Very GREATER", and $BIPI_N^3$ will be regarded as "Very Very GREATER". The outcomes are

summarized in Table 1. Table 1 uses the following abbreviations: "G" for "GREATER", "MLG" for "More or Less GREATER", "VG" for "Very GREATER", and "VVG" for "Very Very GREATER". Also, the degree of similarity found by Eq. (i) is shown by subscript (i) in Table 1.

The similarity among the aforementioned PyFSs must follow the following conditions:

$$S(G, MLG) > S(G, VG) > S(G, VVG) \quad (11)$$

$$S(MLG, G) > S(MLG, VG) > S(MLG, VVG) \quad (12)$$

$$S(VG, VVG) > S(VG, G) > S(VG, MLG) \quad (13)$$

Table 1

The similarities obtained from Eqs. (6)-(8)

	G	MLG	VG	VVG
G	1.0000 ⁽¹¹⁾	0.8485 ⁽¹¹⁾	0.8066 ⁽¹¹⁾	0.7034 ⁽¹¹⁾
	1.0000 ⁽¹²⁾	0.7368 ⁽¹²⁾	0.6758 ⁽¹²⁾	0.5424 ⁽¹²⁾
	1.0000 ⁽¹³⁾	0.7777 ⁽¹³⁾	0.7219 ⁽¹³⁾	0.5941 ⁽¹³⁾
MLG	0.8485 ⁽¹¹⁾	1.0000 ⁽¹¹⁾	0.6858 ⁽¹¹⁾	0.5408 ⁽¹¹⁾
	0.7368 ⁽¹²⁾	1.0000 ⁽¹²⁾	0.5218 ⁽¹²⁾	0.3706 ⁽¹²⁾
	0.7777 ⁽¹³⁾	1.0000 ⁽¹³⁾	0.5736 ⁽¹³⁾	0.4177 ⁽¹³⁾
VG	0.8485 ⁽¹¹⁾	0.6858 ⁽¹¹⁾	1.0000 ⁽¹¹⁾	0.8546 ⁽¹¹⁾
	0.7368 ⁽¹²⁾	0.5218 ⁽¹²⁾	1.0000 ⁽¹²⁾	0.7461 ⁽¹²⁾
	0.7777 ⁽¹³⁾	0.5736 ⁽¹³⁾	1.0000 ⁽¹³⁾	0.7861 ⁽¹³⁾
VVG	0.6922 ⁽¹¹⁾	0.5404 ⁽¹¹⁾	0.8546 ⁽¹¹⁾	1.0000 ⁽¹¹⁾
	0.5292 ⁽¹²⁾	0.3702 ⁽¹²⁾	0.7461 ⁽¹²⁾	1.0000 ⁽¹²⁾
	0.5810 ⁽¹³⁾	0.4172 ⁽¹³⁾	0.7861 ⁽¹³⁾	1.0000 ⁽¹³⁾

As shown in Table 1, the results indicate that the suggested SMs satisfy the conditions of Eqs. (11)–(13). Therefore, the suggested SM is more effective and practical for applications utilizing linguistic variables.

4. Construction of BIPI-TOPSIS

Making decisions is a common aspect of daily life. In MCDM, the goal is to select the best option by ranking all available alternatives. To achieve this, it is essential to establish an effective process that ensures optimal decision-making. In this section, we focus on developing a distance measure between the BIPISs of PyFSs, which allows us to find an effective solution to MCDM by utilizing TOPSIS.

We suggest a clear approach to handle MCDM issues with uncertain weights. A suggested distance measure is calculated between each alternative A_i and both the belief-plausibility Pythagorean fuzzy negative ideal solution (BIPI-PFNIS) and the belief-plausibility Pythagorean fuzzy positive ideal solution (BIPI-PFPIS) separately. Then, the TOPSIS technique is used to rank the alternatives. Step-wise algorithm for the suggested BIPI-TOPSIS is:

Step 1: Formation of belief and plausible decision matrix. Suppose $A = \{A_1, A_2, \dots, A_m\}$ be set of alternatives to criteria $L = \{L_1, L_2, \dots, L_n\}$. Decision matrix for PyFS will be $D = \{\hat{h}_{ij}\}_{m \times n} = \{BI_{ij}, PI_{ij}\}$. The value of $BI_{ij} = \mu_{ij}^2$ shows the degree of belief, while $PI_{ij} = 1 - \nu_{ij}^2$ shows the degree of plausibility. Decision matrix for belief-plausibility can be constructed as:

$$D = \{\hat{h}_{ij}\}_{m \times n} = \begin{Bmatrix} \{BI_{11}, PI_{11}\} & \{BI_{12}, PI_{12}\} & \cdot & \cdot & \cdot & \{BI_{1n}, PI_{1n}\} \\ \{BI_{21}, PI_{21}\} & \{BI_{22}, PI_{22}\} & \cdot & \cdot & \cdot & \{BI_{2n}, PI_{2n}\} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \{BI_{m1}, PI_{m1}\} & \{BI_{m2}, PI_{m2}\} & \cdot & \cdot & \cdot & \{BI_{mn}, PI_{mn}\} \end{Bmatrix} \quad (14)$$

Step 2: Weights of criteria. In the second stage, weights of criteria are calculated. Suppose for the criteria $L_j (j=1, 2, \dots, n)$ weights are w_j with $\sum_{j=1}^n w_j = 1$ for $0 \leq w_j \leq 1$. Since we are unaware of the weights, a new way is suggested to calculate them:

$$w_j = \frac{((3BI_j + PI_j)/2)}{\sum_{j=1}^n ((3BI_j + PI_j)/2)} \quad (15)$$

Step 3: Belief and plausible Pythagorean fuzzy positive and negative ideal solutions. During TOPSIS, the criteria L may be distributed into two disconnected sets; i.e., profit criteria are R_1 and cost criteria are R_2 , where $R_1 \subseteq L$, $R_2 \subseteq L$, and $R_1 \cap R_2 = \emptyset$. According to the TOPSIS algorithm, the belief and plausible positive ideal solution (BIPI-PIS) and belief and plausible negative ideal solution (BIPI-NIS) are:

$$N^+ = \{ \langle L_j, \{BI_j^+, PI_j^+\} \rangle : L_j \in L \} \quad (16)$$

$$N^- = \{ \langle L_j, \{BI_j^-, PI_j^-\} \rangle : L_j \in L \} \quad (17)$$

where $\{BI_j^+, PI_j^+\} = \{1, 1\}$ and $\{BI_j^-, PI_j^-\} = \{0, 0\}$ if $j \in R_1$ and $\{BI_j^+, PI_j^+\} = \{0, 0\}$ and $\{BI_j^-, PI_j^-\} = \{1, 1\}$ if $j \in R_2$.

Step 4: Calculating the distance from each alternative to BIPI-PIS and BIPI-NIS. In this step, we use our proposed distance measure from Eq. (1). From this distance, we construct the distance between all alternatives to BIPI-PIS and BIPI-NIS as:

$$d(A_i, N^+) = F^+(A_i) \quad (18)$$

$$d(A_i, N^-) = F^-(A_i) \quad (19)$$

Step 5: Calculation of relative closeness coefficient and final ranking. In the final step, the alternatives are ranked according to criteria defined for each alternative. Then, a relative closeness $Q(A_i)$ with respect to belief and plausibility distances and SMs is found. The degree of relative closeness coefficient is:

$$Q(A_i) = \frac{F^-(A_i)}{F^-(A_i) + F^+(A_i)} \quad (20)$$

The alternatives are well-arranged according to the degree of closeness. The most favorable alternative is the one having the greatest value of degree of relative closeness.

Example 6. The proposed BIPI-TOPSIS algorithm is implemented to address the material selection problem for solar plate collectors. The global power crisis is a pressing issue characterized by the increasing demand for energy outpacing the supply, leading to widespread electricity shortages, rising costs, and frequent blackouts. This crisis is exacerbated by the heavy reliance on fossil fuels, which are not only depleting rapidly but also contributing significantly to environmental pollution and climate change. As the world grapples with these challenges, solar plate technology emerges as a viable solution. Solar plates, or photovoltaic panels, harness the sun's abundant energy, converting it into electricity without emitting greenhouse gases. This renewable energy source is not only sustainable and environmentally friendly but also increasingly cost-effective due to technological advancements and economies of scale. By integrating solar energy into the global power grid, countries can reduce their dependence on fossil fuels, ensure a more stable and reliable energy supply, and mitigate the adverse environmental impacts of traditional power generation methods. Expanding the use of solar plates, alongside other renewable energy technologies, represents a pivotal strategy in addressing the global power crisis and promoting a sustainable energy future.

Making the best material choice for solar plate collectors is a complex procedure that includes weighing a number of factors to guarantee effectiveness, longevity, affordability, and environmental sustainability. MCDM is a useful technique that offers a methodical framework for ranking and assessing various materials according to a variety of criteria. Several materials are commonly considered for use in solar plate collectors, each with distinct properties, such as copper, aluminum, steel, etc. The aim is to select the best material for solar plate collectors. We have a set of five alternatives and five decision criteria (i.e., "safety level temperature and density", "corrosion resistance", "thermal conductivity", "environmental impact", and "availability"):

- $A_1 = \{(x_1, 0.2, 0.4), (x_2, 0.4, 0.6), (x_3, 0.7, 0.4), (x_4, 0.5, 0.4), (x_5, 0.6, 0.3)\}$,
- $A_2 = \{(x_1, 0.6, 0.5), (x_2, 0.3, 0.4), (x_3, 0.6, 0.4), (x_4, 0.8, 0.1), (x_5, 0.5, 0.4)\}$,
- $A_3 = \{(x_1, 0.6, 0.1), (x_2, 0.8, 0.4), (x_3, 0.4, 0.2), (x_4, 0.7, 0.3), (x_5, 0.4, 0.5)\}$,
- $A_4 = \{(x_1, 0.7, 0.3), (x_2, 0.6, 0.3), (x_3, 0.3, 0.5), (x_4, 0.9, 0.3), (x_5, 0.4, 0.3)\}$, and
- $A_5 = \{(x_1, 0.4, 0.6), (x_2, 0.5, 0.5), (x_3, 0.3, 0.6), (x_4, 0.5, 0.3), (x_5, 0.8, 0.5)\}$.

The corresponding belief and plausible PyFSs are as follows:

- $BIPI_{A_1} = \{(x_1, 0.04, 0.84), (x_2, 0.16, 0.64), (x_3, 0.49, 0.84), (x_4, 0.25, 0.84), (x_5, 0.36, 0.91)\}$,
- $BIPI_{A_2} = \{(x_1, 0.36, 0.75), (x_2, 0.09, 0.84), (x_3, 0.36, 0.84), (x_4, 0.64, 0.99), (x_5, 0.25, 0.84)\}$,
- $BIPI_{A_3} = \{(x_1, 0.36, 0.99), (x_2, 0.64, 0.84), (x_3, 0.16, 0.96), (x_4, 0.49, 0.91), (x_5, 0.16, 0.75)\}$,
- $BIPI_{A_4} = \{(x_1, 0.49, 0.91), (x_2, 0.36, 0.91), (x_3, 0.09, 0.75), (x_4, 0.81, 0.91), (x_5, 0.16, 0.91)\}$, and
- $BIPI_{A_5} = \{(x_1, 0.16, 0.64), (x_2, 0.25, 0.75), (x_3, 0.09, 0.64), (x_4, 0.25, 0.91), (x_5, 0.64, 0.75)\}$.

Step 1. Table 2 shows the belief and plausible decision matrix for the given set of alternatives and criteria.

Table 2

Belief and plausible decision matrix

Alternatives/criteria	M_1	M_2	M_3	M_4	M_5
$BIPI_{A_1}$	(0.04,0.84)	(0.16,0.64)	(0.49,0.84)	(0.25,0.84)	(0.36,0.91)
$BIPI_{A_2}$	(0.36,0.75)	(0.09,0.84)	(0.36,0.84)	(0.64,0.99)	(0.25,0.84)
$BIPI_{A_3}$	(0.36,0.99)	(0.64,0.84)	(0.16,0.96)	(0.49,0.91)	(0.16,0.75)
$BIPI_{A_4}$	(0.49,0.91)	(0.36,0.91)	(0.09,0.75)	(0.81,0.91)	(0.16,0.91)
$BIPI_{A_5}$	(0.16,0.64)	(0.25,0.75)	(0.09,0.64)	(0.25,0.91)	(0.64,0.75)

Step 2. The next step is to calculate criterion weights. By using Eq. (15), we got $w_1 = 0.14$, $w_2 = 0.26$, $w_3 = 0.1$, $w_4 = 0.2$, and $w_5 = 0.3$.

Steps 3-4. The corresponding BIPI-PIS and BIPI-NIS were obtained by using Eq. (16) and Eq. (17), respectively. A weighted distance of each alternative to the belief- and plausibility-based BIPI-PIS and BIPI-NIS was calculated using the proposed distance measure given in Eq. (1). Distance from each alternative to BIPI-PIS and BIPI-NIS were $d(A_1, N^+) = 0.0448$, $d(A_2, N^+) = 0.0429$, $d(A_3, N^+) = 0.0435$, $d(A_4, N^+) = 0.0383$, $d(A_5, N^+) = 0.0426$, $d(A_1, N^-) = 0.0290$, $d(A_2, N^-) = 0.0309$, $d(A_3, N^-) = 0.0302$, $d(A_4, N^-) = 0.0354$, and $d(A_5, N^-) = 0.0312$.

Step 5. In the final step, the best alternative is ranked using the relative closeness coefficient as $Q(A_1) = 0.3931$, $Q(A_2) = 0.4185$, $Q(A_3) = 0.4099$, $Q(A_4) = 0.4802$, and $Q(A_5) = 0.4228$. The alternatives were ranked according to the highest value as $A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$. Accordingly, the alternative A_4 was identified as the most favorable among all options.

5. Conclusion and Future Directions

Previously, researchers have attempted to develop information measures in the literature, but there is still room for improvement and to find new applications in some new directions. Thus, a belief and plausibility measure of PyFSs based on DST is proposed in this study. Membership and non-membership grades for an element in PyFSs are utilized to construct belief and plausibility functions and used to construct the new distance and similarities for PyFSs. These measures are then applied to recognize different patterns and linguistic hedges. Moreover, the proposed distance measure is used to construct the BIPI-TOPSIS. Various numerical examples are given to demonstrate the effectiveness and applicability. According to our analysis, the proposed measures are logical and appropriate in handling uncertainty. Future work will focus on extending the proposed approach to soft sets, bipolar FSs, and neutrosophic FSs, with corresponding applications.

Funding

This research was not funded by any grant.

Conflicts of Interest

The authors declare no conflicts of interest.

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